# Transformers are RNNs:
# Fast Autoregressive Transformers with Linear Attention

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ICML, July 2020

https://linear-transformers.com/

Funded by FNSNF

# Transformers are performant

Transformer models have demonstrated impressive performance on

- NLP (Vaswani et al., 2017; Devlin et al., 2019; Dai et al., 2019; Yang et al., 2019; Radford et al., 2019)
  - Neural Machine Translation
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  - Neural Machine Translation
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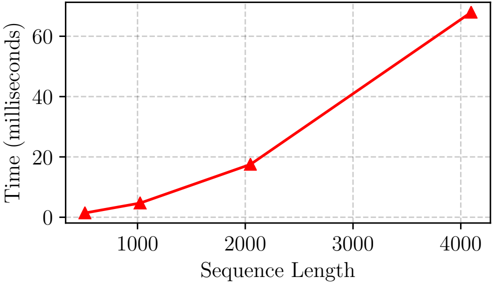
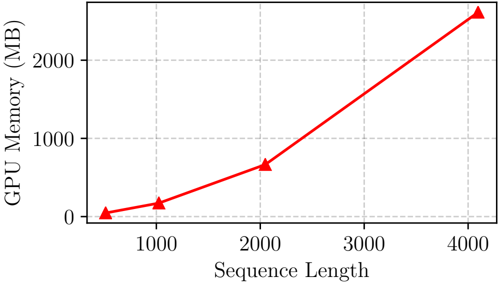
- Speech & audio processing (Sperber et al., 2018)
- Autoregressive image generation and general computer vision (Child et al., 2019; Parmar et al., 2019; Carion et al., 2020; Cordonnier et al., 2020)

# Transformers are hard to scale

Self-attention **computation and memory scales** as $\mathcal{O}\left(N^2\right)$ with respect to the **sequence length**.




A single self-attention layer in an NVIDIA GTX 1080 Ti

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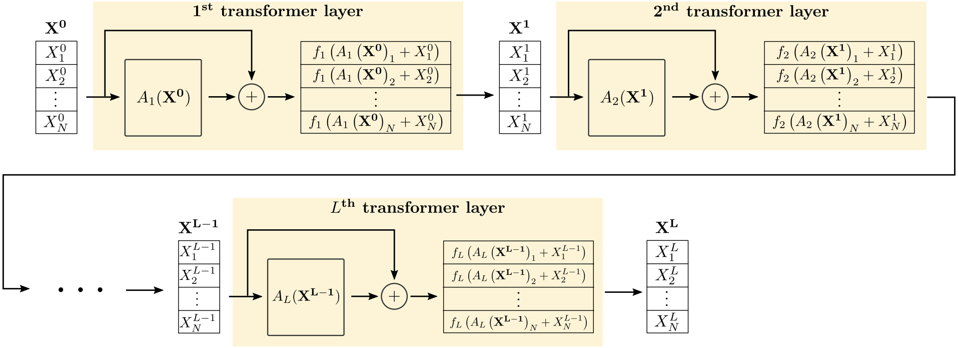
- A transformer model with **linear complexity** both for memory and computation **during training**
- A transformer model with **linear computational complexity and constant memory** for **autoregressive inference**
- Unravel the **relation between transformers and RNNs**

# Definition of a transformer



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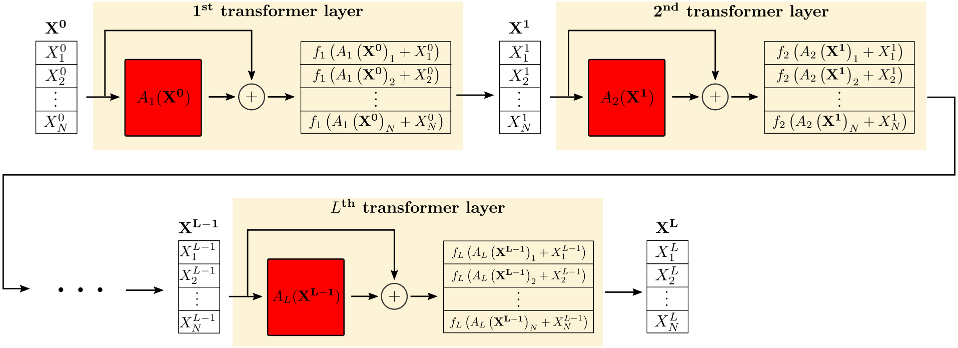

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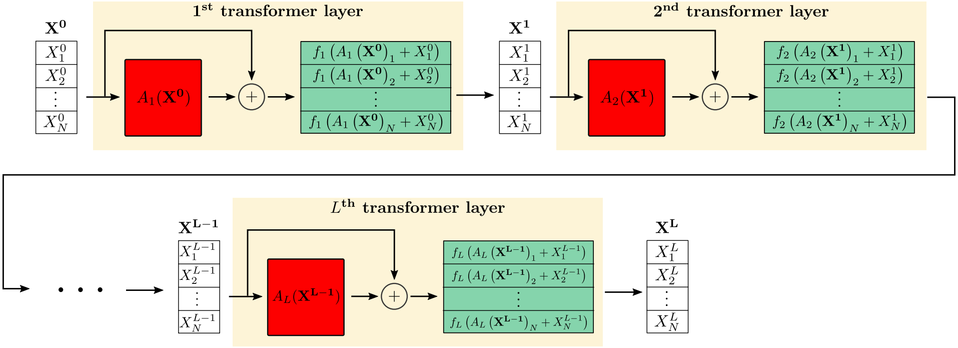

# Self-Attention

The commonly used attention mechanism is the scaled dot product attention

$$Q = XW_Q$$
$$K = XW_K$$
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$$A_l(X) = V' = \text{softmax}\left(\frac{QK^T}{\sqrt{D}}\right)V$$

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Quadratic complexity

# Linear Attention

What if we write the self-attention using an **arbitrary similarity score?**

$$V_i' = \frac{\sum_{j=1}^{N} \mathrm{sim}\left(Q_i, K_j\right) V_j}{\sum_{j=1}^{N} \mathrm{sim}\left(Q_i, K_j\right)}$$

# Linear Attention

What if this similarity is a kernel, namely $\mathrm{sim}(a, b) = \phi(a)^T \phi(b)$?

$$
\begin{aligned}
V_i' &= \frac{\sum_{j=1}^{N} \mathrm{sim}(Q_i, K_j)\, V_j}{\sum_{j=1}^{N} \mathrm{sim}(Q_i, K_j)} \\
&= \frac{\sum_{j=1}^{N} \phi(Q_i)^T \phi(K_j)\, V_j}{\sum_{j=1}^{N} \phi(Q_i)^T \phi(K_j)}
\end{aligned}
$$

Kernelization

# Linear Attention

**Matrix products are associative** which makes the attention computation $\mathcal{O}(N)$ with respect to the sequence length.

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&\qquad\qquad\qquad\qquad \text{Kernelization} \\
&= \frac{\sum_{j=1}^{N} \phi\left(Q_i\right)^T \phi\left(K_j\right) V_j}{\sum_{j=1}^{N} \phi\left(Q_i\right)^T \phi\left(K_j\right)} \\
&\qquad\qquad\qquad\qquad \text{Associativity property} \\
&= \frac{\phi\left(Q_i\right)^T \sum_{j=1}^{N} \phi\left(K_j\right) V_j^T}{\phi\left(Q_i\right)^T \sum_{j=1}^{N} \phi\left(K_j\right)}
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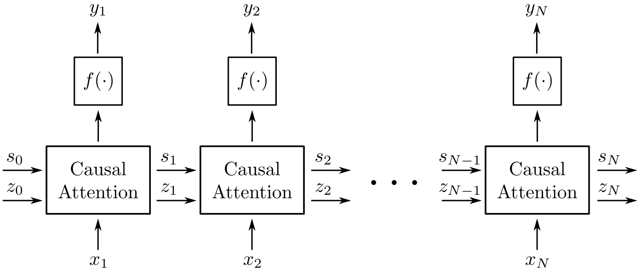
Naive computation of $S_i$ and $Z_i$ results in quadratic complexity.

# Transformers are RNNs

Autoregressive transformers can be written as a function that **receives an input** $x_i$, **modifies the internal state** $\{s_{i-1}, z_{i-1}\}$ and **predicts an output** $y_i$.



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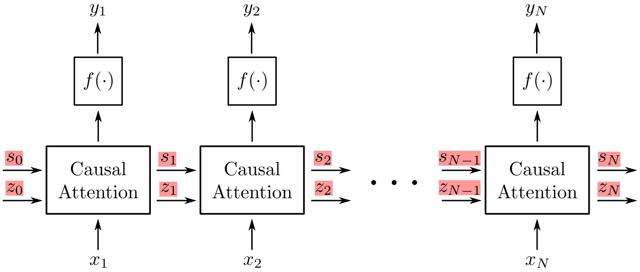

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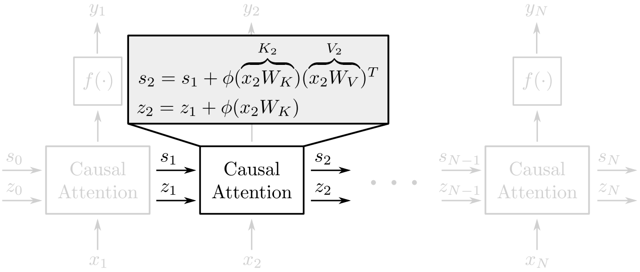

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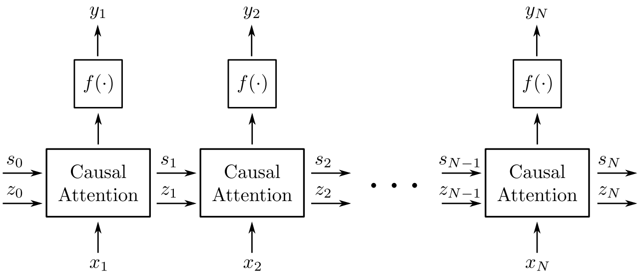


Autoregressive inference with **linear complexity and constant memory**.

# Practical implications

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- We need a simple **finite dimensional feature map** to speed up computation
- We **derive the gradients as cumulative sums** which allows for a significant speed-up

# Experimental setup

Baselines

- Softmax transformer (Vaswani et al., 2017)
- LSH attention from Reformer (Kitaev et al., 2020)

Experiments

- Artificial benchmark for computational and memory requirements
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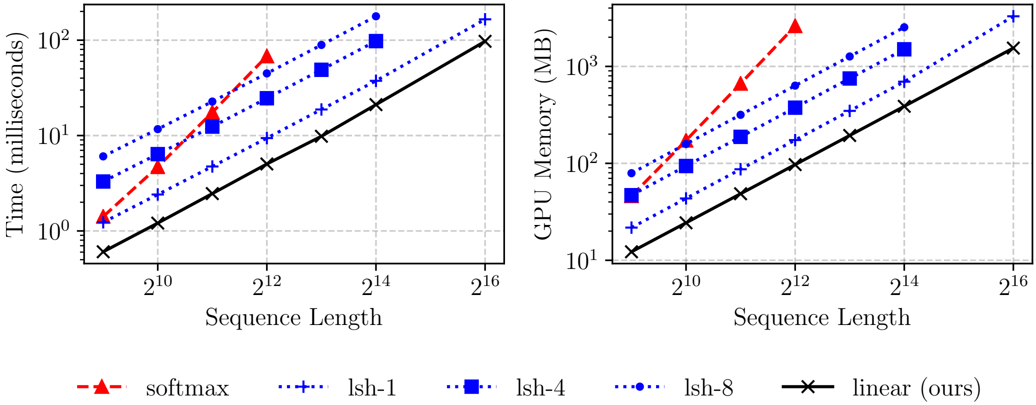
# Benchmark


Time (milliseconds)
$10^0$
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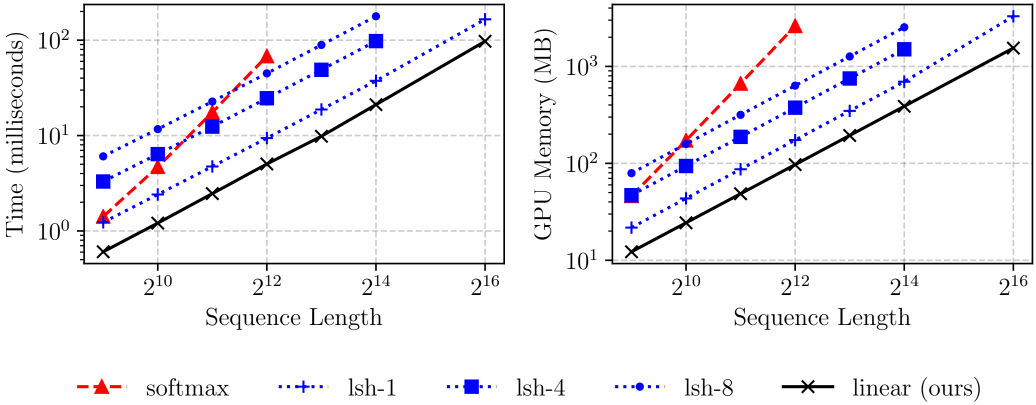

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# Autoregressive image generation

**Unconditional samples after 250 epochs on MNIST**

Ours (0.644 bpd)

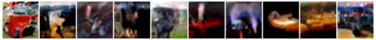
Softmax (0.621 bpd)

LSH-1 (0.745 bpd)

LSH-4 (0.676 bpd)

**Unconditional samples after 1 GPU week on CIFAR-10**

Ours (3.40 bpd)



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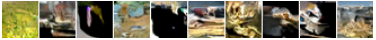

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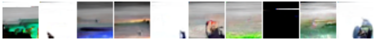

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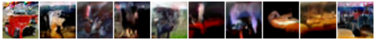

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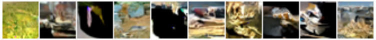

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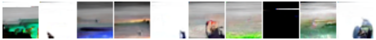

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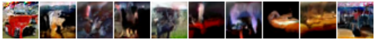

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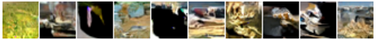

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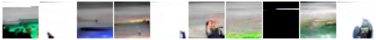

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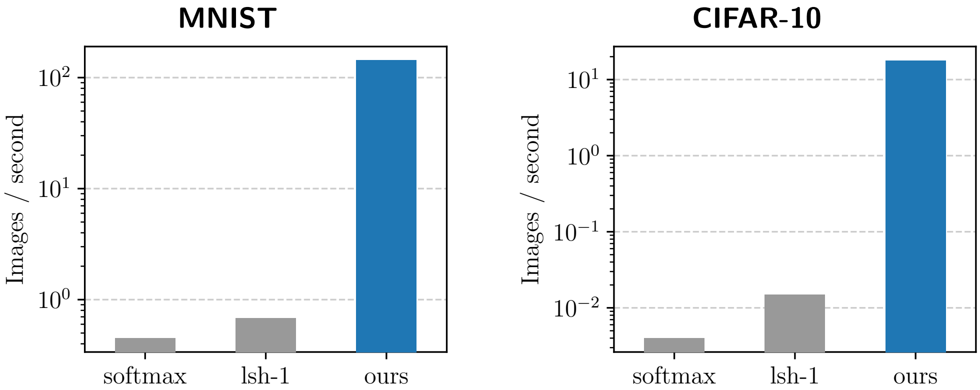

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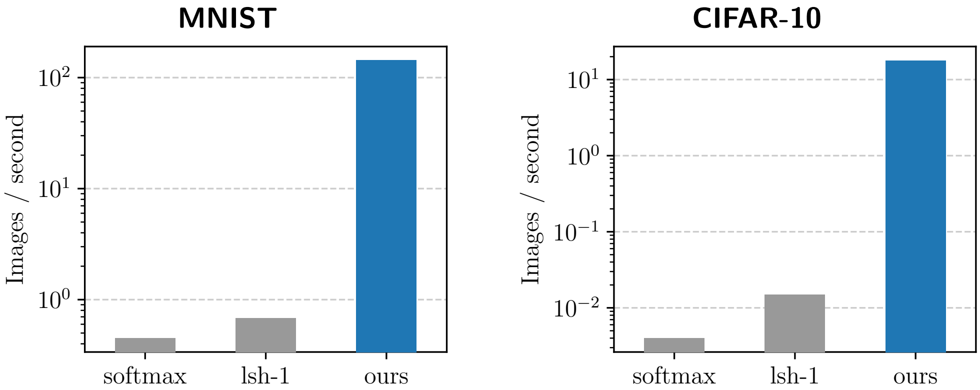

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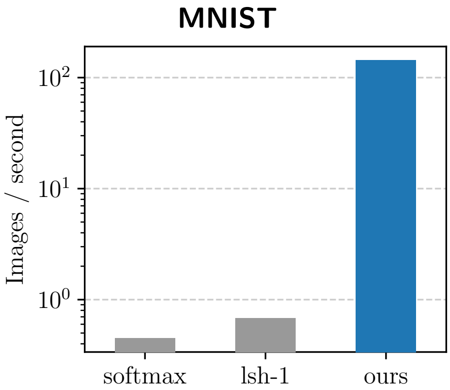
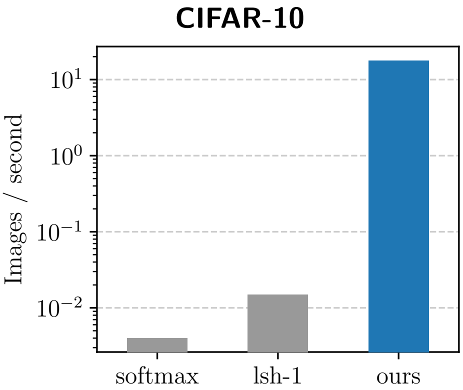
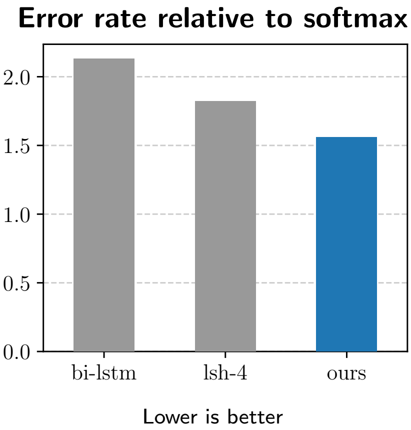
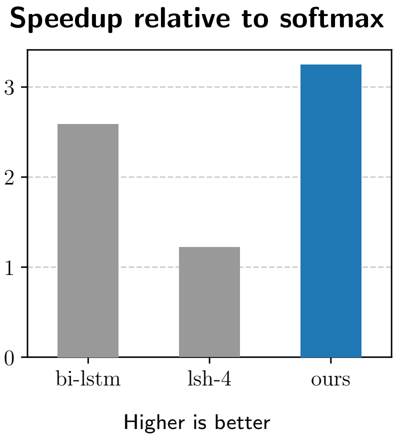


# Automatic speech recognition



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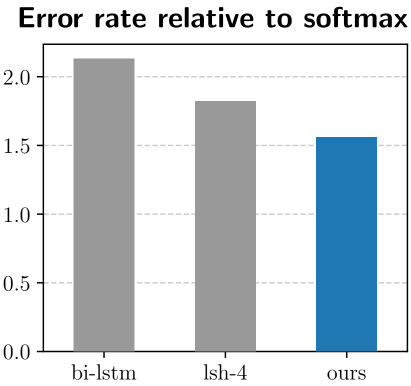


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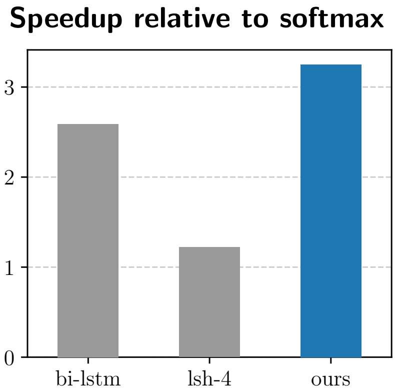


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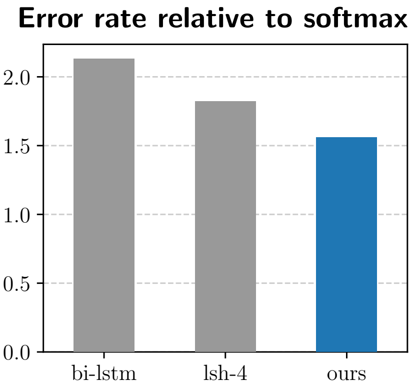


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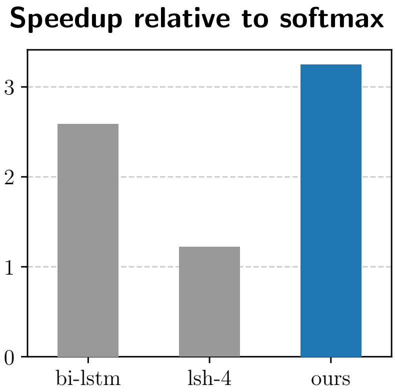


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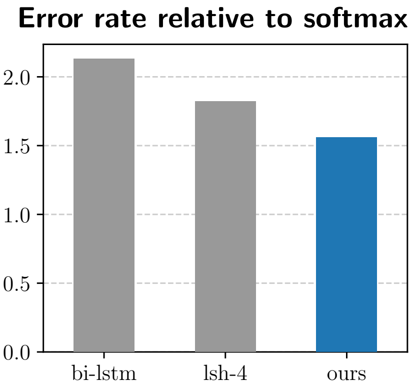


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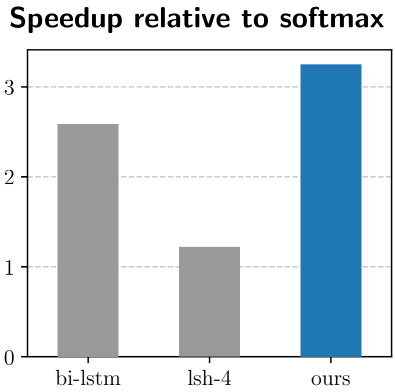


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# Summary

- **Kernel feature maps** and **matrix associativity** yield an attention with linear complexity.
- Computing the key value matrix as a **cumulative sum** extends our efficient attention computation to the autoregressive case
- Using the RNN formulation to perform autoregressive inference requires **constant memory** and is **many times faster**

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# Thank you for your time!

Check out the code at https://linear-transformers.com/ .

```python
from fast_transformers.builders import TransformerEncoderBuilder
linear_bert = TransformerEncoderBuilder.from_kwargs(
    n_layers=12,
    n_heads=12,
    query_dimensions=64,
    value_dimensions=64,
    feed_forward_dimensions=3072,
    attention_type="linear",
).get()
# dummy 4000 long sequence
y = linear_bert(torch.rand(10, 4000, 768))
```

# References I

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